

Thermo Field Dynamics and Kinetic Coefficients of a Charged Boson Gas*

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Abstract

Thermo Field Dynamics for inhomogeneous systems is generalized to quantum fields with a continuous single-particle mass spectrum. The modification of the hamiltonian in states with a local thermal Bogoliubov symmetry is used to calculate thermal conductivity and diffusion coefficient of pions interacting with hot, compressed nuclear matter.

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1 Introduction

Many efforts in contemporary physics are devoted to the study of relativistic quantum systems in non-equilibrium states, like e.g. the expansion of hot, compressed nuclear matter. Unfortunately, the combination of relativistic quantum field theory and statistical mechanics required for the theoretical description of such systems, i.e., relativistic transport theory, is far from well understood and still subject to discussion. As a result, most publications in this field either start from field theory and end up with some transport equation that can hardly be solved – or solve a transport equation, which has lost much of its touch with quantum field theory.

The present paper attempts to bridge this gap through a different approach: The response of a quantum system to gradients in temperature and chemical potential is calculated *directly*, i.e., without the detour of transport theory. To this end a formalism is introduced, which incorporates the space-time dependence of temperature and chemical potential into the quantization rules for the interacting quantum fields.

The paper is organized as follows: A brief introduction to the formalism of Thermo Field Dynamics is followed by its application to a free charged scalar field in global as well as local equilibrium states. The ideas are then extended to the interacting field, and used to calculate the full two-point function for spatially inhomogeneous states approximately.

In section 5, the formal results are then applied to calculate the currents of the system, which arise due to the spatial inhomogeneity. Kinetic (or transport) coefficients are obtained in terms of the spectral function of the interacting field, and compared to formal results obtained with different methods. Numerical results are presented for a simple approximation to pions interacting with a nuclear medium.

An important aspect of finite temperature field theory is the existence of two different, mutually commuting representations of the canonical commutation relations: The Hilbert space of the quantum description is "doubled" with respect to the zero temperature case [1]. For a description in terms of Green's functions, this requires to use 2×2 matrix valued propagators. The two best developed realizations of such a formalism are Thermo Field Dynamics (TFD)[2] and the Closed-Time-Path method [3].

While the vacuum state of quantum field theory is invariant under Lorentz transformations, matter states have, in general, less symmetry. This leads to a modification of the particle spectrum needed for the description of such a state [4]. Hence, a consistent quantum description requires elementary excitations with a continuous mass spectrum, rather than physical quasi-particles of infinite lifetime [1].

Formally this is expressed as spectral function of "particles" which deviates from a δ -function, i.e., which is more than a mass-shell constraint. In several papers it was shown recently, that in the framework of Thermo Field Dynamics a

physical interpretation of the 2×2 matrix structure is obtained also for systems with such a nontrivial single-particle spectrum [5, 6, 7]. In this sense, TFD is superior to the Closed Time-Path method.

In "ordinary" quantum mechanics, a thermal state of a quantum system is described by a statistical operator (or density matrix) W , and the measurement of an observable will yield the average

$$\langle \mathcal{E}(t, \mathbf{x}) \rangle = \frac{\text{Tr} [\mathcal{E}(t, \mathbf{x}) W]}{\text{Tr} [W]}, \quad (1)$$

where the trace is taken over the Hilbert space of the quantum system and \mathcal{E} is the hermitean operator associated with the observable. In Thermo Field Dynamics (TFD), the calculation of this trace is simplified to the calculation of a matrix element

$$\langle \mathcal{E}(t, \mathbf{x}) \rangle = \frac{\langle\langle W^L \| \mathcal{E}(t, \mathbf{x}) \| W^R \rangle\rangle}{\langle\langle W^L \| W^R \rangle\rangle}, \quad (2)$$

with "left" and "right" statistical state defined in terms of the two different commuting representations (see refs. [2, 6] for details).

If gradients of the temperature etc. are present in the state, the hamiltonian of the quantum system, which generates its time evolution, contains a part which is due to the spatial inhomogeneity. The above statistical average in the course of time evolution will be modified by the gradients present in the system.

If the space-dependence of the temperature and chemical potential is sufficiently weak, the response of an observable \mathcal{E} to a gradient is a linear function of the perturbation. We thus express this response $\delta \langle \mathcal{E}(t, \mathbf{x}) \rangle$ up to first order in \hat{H}_1 (the gradient part of the hamiltonian) as

$$\delta \langle \mathcal{E}(t, \mathbf{x}) \rangle = i \int_{t_0}^t d\tau \left\langle \left[\mathcal{E}(t - \tau, \mathbf{x}), \hat{H}_1 \right] \right\rangle. \quad (3)$$

If the commutator can be calculated, one can use it to obtain the transport coefficients of the system, i.e. the coefficients relating the response to the perturbation. The crucial point however is the commutator: Its expectation value is generally calculated as

$$\left\langle \left[\mathcal{E}(t - \tau, \mathbf{x}), \hat{H}_1 \right] \right\rangle = \int_0^\beta d\lambda \left\langle \frac{\partial}{\partial t} \mathcal{E}(t - \tau - i\lambda, \mathbf{x}) \hat{H}_1 \right\rangle, \quad (4)$$

i.e., as the time derivative of a correlation function with *complex time arguments* [8, 9]. The above expression is usually referred to as the Kubo formula [10].

The analytical structure of a correlation function in the complex time plane is related to the properties of its Fourier transform in the complex energy plane.

However, the latter is plagued by pathologies of perturbation theory which are especially severe when considering nuclear phenomena [11, 12]. The most natural way to overcome this problem is to use TFD in a *direct* computation of the commutator expectation value of eqn. (3).

We therefore consider a complex, scalar boson field describing spinless charged excitations in a statistical system not too far from equilibrium. In the spirit of the first remark, one could think of this field as describing positive and negative pions in nuclear matter.

According to the reasoning above, the thermal boson field is then described by two field operators $\phi_x, \tilde{\phi}_x$ and their adjoints $\phi_x^\star, \tilde{\phi}_x^\star$, with canonical commutation relations

$$\begin{aligned} [\phi(t, \mathbf{x}), \partial_t \phi^\star(t, \mathbf{x}')] &= i\delta^3(\mathbf{x} - \mathbf{x}') \\ [\tilde{\phi}(t, \mathbf{x}), \partial_t \tilde{\phi}^\star(t, \mathbf{x}')] &= -i\delta^3(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (5)$$

but commuting with each other. They are combined in a statistical doublet

$$\Phi_x = \begin{pmatrix} \phi_x \\ \tilde{\phi}_x^\star \end{pmatrix}, \quad (6)$$

and the 2×2 matrix valued propagator of this doublet is

$$\begin{aligned} D^{(ab)}(x, x') &= -i \langle \langle W^L \parallel \top \left[\Phi_x \Phi_{x'}^\dagger \right] \parallel W^R \rangle \rangle / \langle \langle W^L \parallel W^R \rangle \rangle \\ &= -i \begin{pmatrix} \langle \langle W^L \parallel \top [\phi_x \phi_{x'}^\star] \parallel W^R \rangle \rangle & \langle \langle W^L \parallel \top [\phi_x \tilde{\phi}_{x'}] \parallel W^R \rangle \rangle \\ \langle \langle W^L \parallel \top [\tilde{\phi}_x^\star \phi_{x'}] \parallel W^R \rangle \rangle & \langle \langle W^L \parallel \top [\tilde{\phi}_x^\star \tilde{\phi}_{x'}] \parallel W^R \rangle \rangle \end{pmatrix} \frac{1}{\langle \langle W^L \parallel W^R \rangle \rangle} \end{aligned} \quad (7)$$

The free as well as the interacting scalar field can be expand into momentum eigenmodes

$$\begin{aligned} \phi_x &= \int \frac{d^3 \mathbf{k}}{\sqrt{(2\pi)^3}} \left(a_{k-}^\dagger(t) e^{-i\mathbf{k}\mathbf{x}} + a_{k+}(t) e^{i\mathbf{k}\mathbf{x}} \right) \\ \tilde{\phi}_x &= \int \frac{d^3 \mathbf{k}}{\sqrt{(2\pi)^3}} \left(\tilde{a}_{k-}^\dagger(t) e^{i\mathbf{k}\mathbf{x}} + \tilde{a}_{k+}(t) e^{-i\mathbf{k}\mathbf{x}} \right). \end{aligned} \quad (8)$$

\mathbf{k} is the three-momentum of the modes, therefore in this notation $a_{k-}^\dagger(t)$ creates a negatively charged excitation with momentum \mathbf{k} , while $a_{k+}(t)$ annihilates a positive charge. Henceforth the two different charges are distinguished by an additional index $l = \pm$ whenever possible.

For the free case the commutation relations of the a -operators at different times are simple, while they are unknown for the interacting fields. Note, that in the above expansion we have used a different energy normalization factor than usual. While this might be a little weird for free fields, it simplifies the calculations for the interacting case.

2 Free charged boson field

The free complex scalar quantum fields are expand as specified in eqn. (8). The operators have commutation relations

$$\begin{aligned}\left[a_{kl}(t), a_{k'l'}^\dagger(t)\right] &= \frac{1}{\sqrt{2\omega_k}} \delta_{ll'} \delta^3(\mathbf{k} - \mathbf{k}') \\ \left[\tilde{a}_{kl}(t), \tilde{a}_{k'l'}^\dagger(t)\right] &= \frac{1}{\sqrt{2\omega_k}} \delta_{ll'} \delta^3(\mathbf{k} - \mathbf{k}') ,\end{aligned}\quad (9)$$

for $l = \pm$. All other commutators vanish, and ω is the free on-shell energy depending on momentum and mass m of the physical particles as

$$\omega_k = \sqrt{\mathbf{k}^2 + m^2} . \quad (10)$$

The free physical states created and annihilated by the above operators are stable. However, they cannot be used as input for perturbative calculations, since an infinitely small interaction gives them a finite lifetime [1].

The stable (albeit unobservable) quasi-particle states are related to the physical states through a Bogoliubov transformation. The hamiltonian of the system is invariant under this transformation, if the system is in global equilibrium. Hence, in this case the system has a global Bogoliubov symmetry. The transformation can be written in matrix form as

$$\begin{aligned}\begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} &= (\mathcal{B}(n_l(\mathbf{k})))^{-1} \begin{pmatrix} \xi_{kl} \\ \tilde{\xi}_{kl}^\dagger \end{pmatrix} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t} \\ \begin{pmatrix} a_{kl}^\dagger(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix}^T &= \begin{pmatrix} \xi_{kl}^\dagger \\ -\tilde{\xi}_{kl} \end{pmatrix}^T \mathcal{B}(n_l(\mathbf{k})) \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k t} .\end{aligned}\quad (11)$$

The ξ -operators create and annihilate stable momentum eigenmodes, and the symbol $\xi^{\dagger\ddagger}$ has been chosen to indicate, that this operator is *not* the hermitean adjoint of ξ . Rather, these operators are thermal conjugate to each other, i.e., they annihilate the thermal ground states according to

$$\xi_{kl} \|W^R\rangle = 0 , \quad \tilde{\xi}_{kl} \|W^R\rangle = 0 , \quad \langle\langle W^L \| \xi_{kl}^\dagger = 0 , \quad \langle\langle W^L \| \tilde{\xi}_{kl}^{\dagger\ddagger} = 0 \quad \forall \mathbf{k}, l = \pm \quad (12)$$

and have commutation relations

$$\begin{aligned}\left[\xi_{kl}, \xi_{k'l'}^{\dagger\ddagger}\right] &= \delta_{ll'} \delta^3(\mathbf{k} - \mathbf{k}') \\ \left[\tilde{\xi}_{kl}, \tilde{\xi}_{k'l'}^{\dagger\ddagger}\right] &= \delta_{ll'} \delta^3(\mathbf{k} - \mathbf{k}')\end{aligned}\quad (13)$$

(all other commutators vanish). Using these rules, any matrix element of field operators can be expressed through the elements of the Bogoliubov matrix.

The Bogoliubov matrix can be parameterized in various ways, leading to unitary inequivalent representations of the thermal quasi-particles [2, 6, 13]. The most useful parameterization however is obtained as linear function of a single parameter n that resembles the phase-space occupation factor,

$$\mathcal{B}(n) = \begin{pmatrix} (1+n) & -n \\ -1 & 1 \end{pmatrix}. \quad (14)$$

In general, the n -parameter can be different for any mode of the system. The modes of the free scalar field are characterized by their momentum, hence $n = n(\mathbf{k})$. The field is complex, implying that the bosons carry a charge and thus are not their own anti-particles. Hence, for the two species of bosons, the n -parameters can be different, $n_+ \neq n_-$. If the charge they carry is a conserved quantity, and the field is in thermal equilibrium, we can chose Bose-Einstein functions as

$$n_{\pm}(\mathbf{k}) = \frac{f_{\pm}(\mathbf{k})}{1 - f_{\pm}(\mathbf{k})} = \frac{1}{e^{\beta(\omega_k \mp \mu)} - 1}, \quad f^{\pm}(\mathbf{k}) = e^{-\beta(\omega_k \mp \mu)}. \quad (15)$$

Note, that in principle a generalized equilibrium state with $\beta = \beta(\mathbf{k})$ could be used as well.

It is an easy task to calculate the propagator from the above considerations, we refer to the existing literature or the next section for its explicit form. More instructive for our purpose is the case, where one uses a Bogoliubov transformation that mixes the momenta as [14]

$$\begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^{\dagger}(t) \end{pmatrix} = \frac{1}{\sqrt{2\omega_k}} \int d^3 \mathbf{q} \left(\tilde{\mathcal{B}}_l^{-1}(\mathbf{q}, \mathbf{k}) \right)^* \begin{pmatrix} \xi_{ql} \\ \tilde{\xi}_{ql}^{\dagger} \end{pmatrix} e^{-i\omega_q t} \\ \begin{pmatrix} a_{kl}^{\dagger}(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix}^T = \frac{1}{\sqrt{2\omega_k}} \int d^3 \mathbf{q} \begin{pmatrix} \xi_{ql}^{\dagger} \\ -\xi_{ql} \end{pmatrix}^T \tilde{\mathcal{B}}_l(\mathbf{q}, \mathbf{k}) e^{i\omega_q t}. \quad (16)$$

This momentum mixing occurs, when one makes the Bogoliubov symmetry mentioned before a *local* symmetry. Such a process is called the *gauging* of the symmetry, and in a quite natural way introduces a coupling to external parameters into the model [15]. The generalized Bogoliubov matrix is expressed as

$$\tilde{\mathcal{B}}_l(\mathbf{q}, \mathbf{k}) = \begin{pmatrix} (\delta^3(\mathbf{q} - \mathbf{k}) + N_l(\mathbf{q}, \mathbf{k})) & -N_l(\mathbf{q}, \mathbf{k}) \\ -\delta^3(\mathbf{q} - \mathbf{k}) & \delta^3(\mathbf{q} - \mathbf{k}) \end{pmatrix}. \quad (17)$$

It obeys the relations

$$\begin{aligned} \left(\tilde{\mathcal{B}}_l^{-1}(\mathbf{q}, \mathbf{k}) \right)^* \tilde{\mathcal{B}}_l(\mathbf{q}, \mathbf{k}') &= \delta^3(\mathbf{k} - \mathbf{q}) \delta^3(\mathbf{q} - \mathbf{k}') \text{diag}(1, 1) \\ &+ T_0 \tau_3 N_l(\mathbf{k}, \mathbf{k}') \left(\delta^3(\mathbf{k} - \mathbf{q}) - \delta^3(\mathbf{q} - \mathbf{k}') \right) \\ \int d^3 \mathbf{q} \left(\tilde{\mathcal{B}}_l^{-1}(\mathbf{q}, \mathbf{k}) \right)^* \tilde{\mathcal{B}}_l(\mathbf{q}, \mathbf{k}') &= \delta^3(\mathbf{k} - \mathbf{k}') \text{diag}(1, 1), \end{aligned} \quad (18)$$

with the Pauli matrix $\tau_3 = \text{diag}(1, -1)$ and

$$T_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \quad (19)$$

The occupation number density parameter appearing here is the Fourier transform of a space-local quantity

$$N_l(\mathbf{q}, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3 \mathbf{z} e^{-i(\mathbf{q}-\mathbf{k})\mathbf{z}} n_l((\mathbf{q} + \mathbf{k})/2, \mathbf{z}) , \quad (20)$$

and we specify that the $n_l(\mathbf{k}, \mathbf{z})$ are real functions. This implies, that $N_l^*(\mathbf{q}, \mathbf{k}) = N_l(\mathbf{k}, \mathbf{q})$. Moreover, the generalized Bogoliubov matrix from eqn. (17) is related to the one from (14) as

$$\tilde{B}_l(\mathbf{q}, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3 \mathbf{z} e^{-i(\mathbf{q}-\mathbf{k})\mathbf{z}} \mathcal{B}(n_l((\mathbf{q} + \mathbf{k})/2, \mathbf{z})) . \quad (21)$$

As one can check within a few lines, the above transformation preserves the canonical commutation relations. However, the time evolution is complicated by the simple fact that the phase factors in eqn. (16) carry the momentum index of the ξ -operators. Indeed, one obtains for the time derivative of the a -operators

$$\begin{aligned} i \frac{\partial}{\partial t} \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} &= \omega_k \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} \\ &+ \int d^3 \mathbf{q} \sqrt{\frac{\omega_q}{\omega_k}} N_l(\mathbf{k}, \mathbf{q}) (\omega_k - \omega_q) T_0 \tau_3 \begin{pmatrix} a_{ql}(t) \\ \tilde{a}_{ql}^\dagger(t) \end{pmatrix} . \end{aligned} \quad (22)$$

A similar relation is obtained for the adjoint operators. Note, that the additional term in this time derivative vanishes, if $N(\mathbf{q}, \mathbf{k})$ is proportional to $\delta^3(\mathbf{q} - \mathbf{k})$. This is the case, when $n(\mathbf{k}, \mathbf{z})$ does not depend on the space coordinate \mathbf{z} , i.e., when the system is translationally invariant.

We can now ask, what kind of Hamilton operator \hat{H} would give a time derivative according to eqn. (22) if used in the Heisenberg equation

$$i \frac{\partial}{\partial t} \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} = \left[\begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix}, \hat{H} \right] . \quad (23)$$

Clearly, such a hamiltonian has two parts: one "bare" giving the unperturbed time evolution

$$\hat{H}_0 = \sum_{l=\pm} \int d^3 \mathbf{k} 2\omega_k^2 \begin{pmatrix} a_{kl}^\dagger(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix}^T \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} , \quad (24)$$

and a part that vanishes for homogeneous systems:

$$\hat{H}_1 = \sum_{l=\pm} \int d^3 \mathbf{k} d^3 \mathbf{q} \begin{pmatrix} a_{kl}^\dagger(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix}^T T_0 \tau_3 \mathcal{H}_l(\mathbf{k}, \mathbf{q}) \begin{pmatrix} a_{ql}(t) \\ \tilde{a}_{ql}^\dagger(t) \end{pmatrix} , \quad (25)$$

with

$$\mathcal{H}_l(\mathbf{k}, \mathbf{q}) = N_l(\mathbf{k}, \mathbf{q}) 2\sqrt{\omega_k \omega_q} (\omega_k - \omega_q) . \quad (26)$$

Note, that the dimension of the numerical factors in (24) and (25) is mass^2 , due to the energy normalization factors contained in the a -operators, cf. eqn. (9). The further discussion of this modified hamiltonian we postpone until the above expression has been generalized to interacting systems.

3 Interacting fields in global equilibrium

The procedure to derive thermal Green's functions for interacting fields has been outlined in refs. [6, 7], and thus will not be repeated here in full detail. It is based on the expansion of the interacting field into modes with definite energy and momentum according to ref. [1]. Their superposition with a positive weight function then gives the fully interacting field.

Such an expansion is necessary, because at finite density and temperature the irreducible representations of the space-time symmetry group are characterized by two continuous parameters [4]. Since these fields contain the full interaction of the system, the commutation relations of the fields with definite energy and momentum are not known in general. However, for the task of calculating two-point functions or bilinear expectation values, one needs to know only the expectation value of such commutators – and this information can be absorbed into the weight function.

In other words, for the calculation of bilinear expectation values of interacting fields it is sufficient to consider the modes of definite energy and momentum as generalized free fields [1]. The full information about the single-particle spectrum of the theory is contained in the weight functions.

The final step in deriving full propagators at finite temperature then consists in applying the thermal quasi-particle picture to each of these modes. We thus write for the operators of the field expansion

$$\begin{pmatrix} a_{k\pm}(t) \\ \tilde{a}_{k\pm}^\dagger(t) \end{pmatrix} = \int_0^\infty dE \rho_\pm^{1/2}(E, \mathbf{k}) (\mathcal{B}(n_\pm(E, \mathbf{k})))^{-1} \begin{pmatrix} \xi_{Ek\pm} \\ \tilde{\xi}_{Ek\pm}^\dagger \end{pmatrix} e^{-iEt}$$

$$\begin{pmatrix} a_{k\pm}^\dagger(t) \\ -\tilde{a}_{k\pm}(t) \end{pmatrix}^T = \int_0^\infty dE \rho_\pm^{1/2}(E, \mathbf{k}) \begin{pmatrix} \xi_{Ek\pm}^\dagger \\ -\tilde{\xi}_{Ek\pm} \end{pmatrix}^T \mathcal{B}(n_\pm(E, \mathbf{k})) e^{iEt} . \quad (27)$$

This Bogoliubov transformation for interacting systems defines stable, albeit non-observable quasi-particles [5, 6, 7], created and annihilated by the ξ -operators. The ξ -operators have different commutation relations, cf. ref. [1]

$$\begin{aligned} [\xi_{Ekl}, \xi_{E'k'l'}^\dagger] &= \delta_{ll'} \delta(E - E') \delta^3(\mathbf{k} - \mathbf{k}') \\ [\tilde{\xi}_{Ekl}, \tilde{\xi}_{E'k'l'}^\dagger] &= \delta_{ll'} \delta(E - E') \delta^3(\mathbf{k} - \mathbf{k}') \end{aligned} \quad (28)$$

(all other commutators vanish), but the thermal state conditions remain similar to (12)

$$\xi_{Ekl}\|W^R\rangle\rangle = 0 \quad , \quad \widetilde{\xi}_{Ekl}\|W^R\rangle\rangle = 0 \quad , \quad \langle\langle W^L|\xi_{Ekl}^\dagger = 0 \quad , \quad \langle\langle W^L|\widetilde{\xi}_{Ekl}^\dagger = 0 \quad \forall E, \mathbf{k}, l = \pm \quad . \quad (29)$$

With these rules, all bilinear expectation values can be calculated. Matrix elements of higher powers of interacting field operators however are obtained only approximately, they have a perturbative expansion in terms of the weight functions $\rho_l(E, \mathbf{k})$ [1].

n is a Bose-Einstein distribution function, which for special case of a global equilibrium depends only on the energy of the modes:

$$n_\pm(E, \mathbf{k}) = \frac{f_\pm(E)}{1 - f_\pm(E)} = \frac{1}{e^{\beta(E \mp \mu)} - 1} \quad , \quad f_\pm(E) = e^{-\beta(E \mp \mu)} \quad . \quad (30)$$

In our notation, the weight functions $\rho_l(E, \mathbf{k})$ have support only for positive energies, and their integrals are

$$\begin{aligned} \int_0^\infty dE E \rho_l(E, \mathbf{k}) &= \frac{1}{2} \\ \int_0^\infty dE \rho_l(E, \mathbf{k}) &= Z_{kl} \quad . \end{aligned} \quad (31)$$

Note, that the existence of this spectral decomposition is only guaranteed in case the system is space-time translation invariant, i.e., if it is in a thermal equilibrium state. It follows from these definitions, that

$$\begin{aligned} [a_{kl}(t), a_{k'l'}^\dagger(t)] &= Z_{kl} \delta_{ll'} \delta^3(\mathbf{k} - \mathbf{k}') \\ [\widetilde{a}_{kl}(t), \widetilde{a}_{k'l'}^\dagger(t)] &= Z_{kl} \delta_{ll'} \delta^3(\mathbf{k} - \mathbf{k}') \end{aligned} \quad (32)$$

in generalization of eqn. (9). Collecting these definitions, the full bosonic two-point Green's function, or propagator, at finite temperature is derived as [6, 7]

$$\begin{aligned} D^{(ab)}(k_0, \mathbf{k}) &= \\ &\int_0^\infty dE \rho_+(E, \mathbf{k}) (\mathcal{B}(n_+(E, \mathbf{k})))^{-1} \begin{pmatrix} \frac{1}{k_0 - E + i\epsilon} & \\ & \frac{1}{k_0 - E - i\epsilon} \end{pmatrix} \mathcal{B}(n_+(E, \mathbf{k})) \tau_3 \\ &- \int_0^\infty dE \rho_-(E, \mathbf{k}) \tau_3 \mathcal{B}^T(n_-(E, \mathbf{k})) \begin{pmatrix} \frac{1}{k_0 + E - i\epsilon} & \\ & \frac{1}{k_0 + E + i\epsilon} \end{pmatrix} (\mathcal{B}^T(n_-(E, \mathbf{k}))) \end{aligned} \quad (33)$$

In the above equation, the terms propagating particle and anti-particle states have been kept separately. Eqn. (30) can be continued to negative energy arguments, and

$$n_-(E, \mathbf{k}) = -(1 + n_+(-E, -\mathbf{k})) . \quad (34)$$

Setting $n_+(E, \mathbf{k}) \equiv n(E)$ in the following, the above propagator simplifies to

$$D^{(ab)}(k_0, \mathbf{k}) = \int_{-\infty}^{\infty} dE \mathcal{A}(E, \mathbf{k}) \times (\mathcal{B}(n(E)))^{-1} \begin{pmatrix} \frac{1}{k_0 - E + i\epsilon} & \\ & \frac{1}{k_0 - E - i\epsilon} \end{pmatrix} \mathcal{B}(n(E)) \tau_3 \quad (35)$$

$\mathcal{A}(E, \mathbf{k})$ is the spectral function of the boson field,

$$\mathcal{A}(E, \mathbf{k}) = \rho_+(E, \mathbf{k})\Theta(E) - \rho_-(-E, -\mathbf{k})\Theta(-E) , \quad (36)$$

and the limit of free particles with mass m is recovered when

$$\mathcal{A}(E, \mathbf{k}) \longrightarrow \text{sign}(E) \delta(E^2 - \mathbf{k}^2 - m^2) = \text{sign}(E) \delta(E^2 - \omega_k^2) . \quad (37)$$

4 Interacting fields with local Bogoliubov symmetry

As was stated above, the existence of a spectral decomposition is bound to the equilibrium property of the system, i.e., to its space-time translation invariance [2]. We can expect however, that close to equilibrium the field properties do not change very much. Thus, the considerations from the second section of the present paper can be carried over to the interacting case: We want to study the quantum system under the influence of small gradients in the distribution function n , but with the approximation of a space-time independent spectral weight function $\rho_l(E, \mathbf{k})$. In generalization of eqn. (16):

$$\begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} = \int_0^\infty dE \int d^3\mathbf{q} \rho_l^{1/2}(E, (\mathbf{q} + \mathbf{k})/2) \left(\tilde{\mathcal{B}}_l^{-1}(E, \mathbf{q}, \mathbf{k}) \right)^* \begin{pmatrix} \xi_{Eql} \\ \tilde{\xi}_{Eql}^\dagger \end{pmatrix} e^{-iEt} \\ \begin{pmatrix} a_{kl}^\dagger(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix}^T = \int_0^\infty dE \int d^3\mathbf{q} \rho_l^{1/2}(E, (\mathbf{q} + \mathbf{k})/2) \begin{pmatrix} \xi_{Eql}^\dagger \\ -\tilde{\xi}_{Eql} \end{pmatrix}^T \tilde{\mathcal{B}}_l(E, \mathbf{q}, \mathbf{k}) e^{iEt} \quad (38)$$

The transformation matrix here has the same functional form as in eqn. (17), but now its occupation number density parameter also depends on the energy variable

$$N_l(E, \mathbf{q}, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{z} e^{-i(\mathbf{q}-\mathbf{k})\mathbf{z}} n_l(E, (\mathbf{q} + \mathbf{k})/2, \mathbf{z}) . \quad (39)$$

The propagator matrix according to (7) is calculated as

$$\begin{aligned}
D^{(ab)}(t, \mathbf{x}; t', \mathbf{x}') &= -i \int_0^\infty dE \int \frac{d^3 \mathbf{k} d^3 \mathbf{q} d^3 \mathbf{k}'}{(2\pi)^3} \rho_+^{1/2}(E, (\mathbf{k} + \mathbf{q})/2) \rho_+^{1/2}(E, (\mathbf{q} + \mathbf{k}')/2) \times \\
&(\tilde{\mathcal{B}}_+^{-1}(E, \mathbf{q}, \mathbf{k}))^* \begin{pmatrix} \Theta(t - t') & \\ & -\Theta(t' - t) \end{pmatrix} \tilde{\mathcal{B}}_+(E, \mathbf{q}, \mathbf{k}') \tau_3 e^{-iE(t-t')} e^{i\mathbf{k}\mathbf{x} - i\mathbf{k}'\mathbf{x}'} \\
&- i \int_0^\infty dE \int \frac{d^3 \mathbf{k} d^3 \mathbf{q} d^3 \mathbf{k}'}{(2\pi)^3} \rho_-^{1/2}(E, (\mathbf{k} + \mathbf{q})/2) \rho_-^{1/2}(E, (\mathbf{q} + \mathbf{k}')/2) \times \\
&\tau_3 \tilde{\mathcal{B}}_-^T(E, \mathbf{q}, \mathbf{k}) \begin{pmatrix} \Theta(t' - t) & \\ & -\Theta(t - t') \end{pmatrix} (\tilde{\mathcal{B}}_-^T(E, \mathbf{q}, \mathbf{k}'))^{*-1} e^{iE(t-t')} e^{-i\mathbf{k}\mathbf{x} + i\mathbf{k}'\mathbf{x}'} \quad (40)
\end{aligned}$$

Similar to the equilibrium case and to the time-dependent case one can perform the energy integration because of the linearity of this expression in n . This allows to write the full propagator in terms of Bogoliubov matrices, which depend on space-local distribution functions. The principles of this method are outlined in ref. [6], and will not be discussed in the present paper.

As the last point in considering the above full Green function, consider the imaginary part of the retarded propagator obtained as

$$\begin{aligned}
\text{Im} (D^R(t, \mathbf{x}; t', \mathbf{x}')) &= \text{Im} (D^{(11)}(t, \mathbf{x}; t', \mathbf{x}') - D^{(12)}(t, \mathbf{x}; t', \mathbf{x}')) \\
&= -\pi \int \frac{dE d^3 \mathbf{k}}{(2\pi)^4} e^{-iE(t-t') + i\mathbf{k}(\mathbf{x} - \mathbf{x}')} (\rho_+(E, \mathbf{k}) - \rho_-(-E, -\mathbf{k})) \quad (41)
\end{aligned}$$

The retarded and advanced propagator hence only depend on the coordinate differences, and their imaginary part still gives the spectral function from eqn. (36). This is the only approximation made so far, but it can be shown easily by solving the matrix-valued Schwinger-Dyson equation in coordinate representation, that corrections to the above spectral functions are *of second order in the gradients*.

In generalization of eqn. (22), one thus obtains for the time derivative

$$\begin{aligned}
i \frac{\partial}{\partial t} \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} &= \Omega_{kl} \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}^\dagger(t) \end{pmatrix} \\
&+ \int_0^\infty dE \int d^3 \mathbf{q} \frac{\rho_l^{1/2}(E, (\mathbf{k} + \mathbf{q})/2)}{Z_{ql}} N_l(E, \mathbf{k}, \mathbf{q}) \times \\
&\left(E \rho_l^{1/2}(E, \mathbf{k}) - E \rho_l^{1/2}(E, \mathbf{q}) \right) T_0 \tau_3 \begin{pmatrix} a_{ql}(t) \\ \tilde{a}_{ql}^\dagger(t) \end{pmatrix}. \quad (42)
\end{aligned}$$

Here, the energy Ω is defined as

$$\Omega_{kl} = \frac{1}{Z_{kl}} \int_0^\infty dE E \rho_l(E, \mathbf{k}) = \frac{1}{2Z_{kl}}, \quad (43)$$

and Z_{kl} as in eqn. (31). Obviously, as for the free field, the additional term in this time evolution vanishes for translationally invariant n , i.e., when $N(E, \mathbf{k}, \mathbf{q})$ is proportional to $\delta^3(\mathbf{k} - \mathbf{q})$. It has a generator \hat{H}_1 , which has the same functional form as in eqn. (25), but the kernel reads

$$\mathcal{H}_l(\mathbf{k}, \mathbf{q}) = \int_0^\infty dE \left\{ \frac{\rho_l^{1/2}(E, (\mathbf{k} + \mathbf{q})/2)}{Z_{kl} Z_{ql}} \right\} N_l(E, \mathbf{k}, \mathbf{q}) \left(E \rho_l^{1/2}(E, \mathbf{k}) - E \rho_l^{1/2}(E, \mathbf{q}) \right). \quad (44)$$

The perturbation of the system has a very special form given in eqn. (25). If it is expressed in terms of the ξ -operators, one may use

$$\tilde{\mathcal{B}}(E, \mathbf{q}, \mathbf{q}') \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\tilde{\mathcal{B}}^{-1}(E, \mathbf{k}, \mathbf{k}'))^* = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \delta^3(\mathbf{q} - \mathbf{q}') \delta^3(\mathbf{k} - \mathbf{k}') \quad (45)$$

to see that only the combination $\xi^{\dagger\dagger} \tilde{\xi}^{\dagger\dagger}$ survives:

$$\hat{H}_1 = \sum_{l=\pm} \int_0^\infty dE dE' \int d^3\mathbf{k} d^3\mathbf{k}' \rho_l^{1/2}(E, \mathbf{k}) \rho_l^{1/2}(E', \mathbf{k}') \mathcal{H}_l(\mathbf{k}, \mathbf{k}') \xi_{Ekl}^{\dagger\dagger} \tilde{\xi}_{E'k'l}^{\dagger\dagger}. \quad (46)$$

This simple expression for the gradient part of the hamiltonian can be inserted into eqn. (3). Note, that the total hamiltonian is diagonal in the ξ -operators, but not if expressed in terms of the physical particle operators a, a^\dagger .

5 Transport properties of an interacting scalar field

Observables are expressed as functionals of the interacting fields – and therefore as functionals of the operators a, a^\dagger . Hence, although the state we consider is stationary in terms of the basis defined by the ξ -operators, momentum mixing as introduced above will result in a non-trivial time evolution of *physical* quantities, according to the Heisenberg equation (23). Thus, equation (4) can be used to calculate relevant contributions to system parameters due to the spatial dependence of $n(E, \mathbf{k}, \mathbf{x})$.

Two quantities we are interested in are the conserved particle current operator of the complex scalar field and its expectation value,

$$\hat{\mathbf{j}}(t, \mathbf{x}) = i (\phi_x^* \nabla \phi_x - \phi_x \nabla \phi_x^*) \quad \mathbf{j}(t, \mathbf{x}) = \delta \left\langle \hat{\mathbf{j}}(t, \mathbf{x}) \right\rangle \quad (47)$$

and the conserved energy current

$$\widehat{\mathbf{E}}(t, \mathbf{x}) = (\partial_t \phi_x^* \nabla \phi_x + \partial_t \phi_x \nabla \phi_x^*) \quad \mathbf{E}(t, \mathbf{x}) = \delta \left\langle \widehat{\mathbf{E}}(t, \mathbf{x}) \right\rangle. \quad (48)$$

Inserting these into eqn. (3) then yields an expression for the commutator, which contains *four* field operators instead of *two*. The expectation value of this commutator therefore cannot be expressed completely by the interacting two-point functions we have calculated. For these, it had been explicitly assumed above, that the fields can be written as generalized free fields.

However, the four-field expectation value has a diagrammatic expansion in terms of the propagators we derived - and the lowest order term of this expansion already contains the full single-particle spectrum. We expect, that this lowest order contributes the dominant part of the system's response.

Using the commutation rules for the generalized free fields, e.g. the Bogoliubov transformation and (28), therefore amounts to the neglect of higher correlations. These however can be incorporated through the appropriate Feynman rules [1, 16].

If we insert the expansion of the fields according to (8), all kinds of binary products of the a -operators appear in these operator expressions. However, in the present formulation, out of 4×8 combinations of a -operators contained in a naive calculation of the commutator (4), only two survive due to the special form (46) of \widehat{H}_1 . These are

$$\begin{aligned} & \left\langle \left[a_{kl}^\dagger(t-\tau) a_{k'l}(t-\tau), \xi_{E'q'l}^\dagger \widetilde{\xi}_{Eq'l}^\dagger \right] \right\rangle = \\ & = \rho_l^{1/2}(E, \mathbf{q}) \rho_l^{1/2}(E', \mathbf{q}') \delta^3(\mathbf{k} - \mathbf{q}) \delta^3(\mathbf{k}' - \mathbf{q}') e^{i(E-E')(t-\tau)} \\ & \left\langle \left[\partial_t a_{kl}^\dagger(t-\tau) a_{k'l}(t-\tau), \xi_{E'q'l}^\dagger \widetilde{\xi}_{Eq'l}^\dagger \right] \right\rangle = \\ & = iE \rho_l^{1/2}(E, \mathbf{q}) \rho_l^{1/2}(E', \mathbf{q}') \delta^3(\mathbf{k} - \mathbf{q}) \delta^3(\mathbf{k}' - \mathbf{q}') e^{i(E-E')(t-\tau)} \end{aligned} \quad (49)$$

for $l = \pm$, i.e., both charge species.

We can then assemble the results for the currents, by performing the time integration as if $t \rightarrow \infty$ and $t_0 \rightarrow -\infty$. Alternatively, one could first process the momentum integrations and then pick out the real part of the current. One obtains

$$\mathbf{j}_l(t, \mathbf{x}) = -2\pi i \int_0^\infty dE \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^3} \rho_l(E, \mathbf{k}) \rho_l(E, \mathbf{k}') e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}} (\mathbf{k} + \mathbf{k}') \mathcal{H}_l(\mathbf{k}, \mathbf{k}') \quad (50)$$

for each species. The energy current is formally quite similar,

$$\mathbf{E}_l(t, \mathbf{x}) = -2\pi i \int_0^\infty dE \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^3} \rho_l(E, \mathbf{k}) \rho_l(E, \mathbf{k}') e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}} E \left(\mathbf{k} + \mathbf{k}' \right) \mathcal{H}_l(\mathbf{k}, \mathbf{k}') \quad (51)$$

and the total currents are for both cases

$$\begin{aligned} \mathbf{j}(t, \mathbf{x}) &= \mathbf{j}_+(t, \mathbf{x}) - \mathbf{j}_-(t, \mathbf{x}) \\ \mathbf{E}(t, \mathbf{x}) &= \mathbf{E}_+(t, \mathbf{x}) + \mathbf{E}_-(t, \mathbf{x}) . \end{aligned} \quad (52)$$

In principle the above expressions can be calculated, when the spectral functions and the space-dependence of n are given. However, it is more instructive to perform a gradient expansion for the occupation number density n . For this we use the abbreviation $\mathbf{Q} = (\mathbf{k} + \mathbf{k}')/2$, and expand all momenta around this value, e.g.

$$\rho_l^{1/2}(E, \mathbf{Q}) \left(E \rho_l^{1/2}(E, \mathbf{k}) - E \rho_l^{1/2}(E, \mathbf{k}') \right) \approx (\mathbf{k} - \mathbf{k}') \frac{E}{2} \frac{\partial}{\partial \mathbf{Q}} \rho_l(E, \mathbf{Q}) . \quad (53)$$

The factor $(\mathbf{k} - \mathbf{k}')$ can be taken out of the integrations, as $-i\partial/\partial \mathbf{x} \equiv -i\nabla_{\mathbf{x}}$. Other contributions are of zeroth and second order in this derivative, e.g.

$$\rho_l(E, \mathbf{k}) \rho_l(E, \mathbf{k}') = \rho_l^2(E, \mathbf{Q}) + \mathcal{O} \left((\mathbf{k} - \mathbf{k}')^2 \right) . \quad (54)$$

Therefore, the i -th vector component of the l -charged currents generated by the inhomogeneity of the system is

$$\begin{aligned} \mathbf{j}_l^{(i)}(t, \mathbf{x}) &= 2\pi \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{\mathbf{Q}^{(i)}}{2Z_{Ql}^2} \int dE (\rho_l(E, \mathbf{Q}))^2 \times \\ &\quad \int dE' E' \left\{ \frac{\partial n_l(E', \mathbf{Q}, \mathbf{x})}{\partial \mathbf{x}^{(j)}} \frac{\partial \rho_l(E', \mathbf{Q})}{\partial \mathbf{Q}^{(j)}} \right\} \\ \mathbf{E}_l^{(i)}(t, \mathbf{x}) &= 2\pi \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{\mathbf{Q}^{(i)}}{2Z_{Ql}^2} \int dE E (\rho_l(E, \mathbf{Q}))^2 \times \\ &\quad \int dE' E' \left\{ \frac{\partial n_l(E', \mathbf{Q}, \mathbf{x})}{\partial \mathbf{x}^{(j)}} \frac{\partial \rho_l(E', \mathbf{Q})}{\partial \mathbf{Q}^{(j)}} \right\} \end{aligned} \quad (55)$$

Before this is evaluated further, note that the expression in the curly brackets is nothing but the first term of a full gradient expansion of the product of n and ρ , i.e., the Poisson bracket. It is therefore clear, that the above expression is useful also for situations, where the space-dependence of ρ cannot be neglected: the Poisson bracket then also has a contribution with a space-derivative acting on ρ and a momentum derivative acting on n . The above expression is also

interesting in view of the original equation (3), since the Poisson bracket is the analogon of the commutator, i.e., it contributes a factor \hbar to the the current.

To proceed, some remarks on the derivatives occuring in the Poisson bracket are necessary. It was assumed, that the system is in a local equilibrium state, close to global equilibrium such that the gradients are small and currents are only due to these gradients. This implies, that the spectral function depends only on $Q = |\mathbf{Q}|$, and thus the momentum derivative gives

$$\frac{\partial \rho_l(E', \mathbf{Q})}{\partial \mathbf{Q}^{(j)}} = \frac{\mathbf{Q}^{(j)}}{Q} \frac{\partial \rho_l(E', Q)}{\partial Q} \quad (56)$$

The currents from eqn. (55) thus are symmetric in the indices i and j .

Using as local equilibrium distribution function instead of eqn. (30) the expression

$$n_{\pm}(E, \mathbf{Q}, \mathbf{x}) = \frac{f_{\pm}(E, \mathbf{x})}{1 - f_{\pm}(E, \mathbf{x})} = \frac{1}{e^{\beta(\mathbf{x})(E \mp \mu(\mathbf{x}))} - 1} \quad , \quad f_{\pm}(E, \mathbf{x}) = e^{-\beta(\mathbf{x})(E \mp \mu(\mathbf{x}))} \quad , \quad (57)$$

the derivative with respect to \mathbf{x} has two components,

$$\frac{\partial}{\partial \mathbf{x}^{(j)}} n_{\pm}(E, \mathbf{Q}, \mathbf{x}) = \frac{1}{4 \sinh^2(\beta(E \mp \mu)/2)} \left(\frac{E \mp \mu}{T} \frac{\nabla_x^{(j)} T}{T} \pm \frac{1}{T} \nabla_x^{(j)} \mu \right) . \quad (58)$$

This derivative however has to be taken with care: In eqns. (55), the \mathbf{x} -derivative was shifted through the integrals as if they were all properly defined. This requires, in the absence of a Bose condensate, that the spectral function vanishes linearly at $E = \mu$. A discussion of this requirement will be attempted in a separate publication, here we will take it for granted.

However, this violates the assumption of a spectral function independent of the gradients, since the \mathbf{x} -derivative then also acts on $\rho(E, \mathbf{k})$. To avoid this complication, we define the integral over the distribution function as a principal value integral, i.e., it is only defined in the sense of first calculating

$$I_{Ql} = \int_0^{\infty} dE \rho(E, \mathbf{Q}) n(E, \mathbf{Q}, \mathbf{x}) = \lim_{\epsilon \rightarrow 0} \left[\int_0^{\mu - \epsilon} + \int_{\mu + \epsilon}^{\infty} \right] dE \rho(E, \mathbf{Q}) n(E, \mathbf{Q}, \mathbf{x}) \quad (59)$$

and then taking the \mathbf{x} -derivative. The same decomposition into T and μ -gradient terms then holds for the currents, i.e., with transport coefficients L_{ij} we obtain [9]

$$\begin{aligned} \mathbf{j}_l(t, \mathbf{x}) &= -L_{11} \nabla \mu - L_{12} \frac{\nabla T}{T} \\ \mathbf{E}_l(t, \mathbf{x}) &= -L_{21} \nabla \mu - L_{22} \frac{\nabla T}{T} . \end{aligned} \quad (60)$$

Note, that in our picture L_{12} and L_{21} can be different, i.e. the Onsager relation $L_{ij} = L_{ji}$ is not necessarily fulfilled. The reason for this is clear: in contrast to ordinary many-body quantum physics, the present formulation exhibits dissipation already on the tree-graph level [1]. In other words, the formulation of an interacting theory with continuous spectral functions is *not* micro-reversible, physical states have a finite lifetime.

In summary of the above reasoning, we can thus write the kinetic coefficients with the abbreviations

$$\begin{aligned} X_{Ql} &= \int dE (\rho_l(E, \mathbf{Q}))^2 \\ Y_{Ql} &= \int dE E (\rho_l(E, \mathbf{Q}))^2, \end{aligned} \quad (61)$$

plus the definitions of I_{Ql} from eqn. (59), Z_{Ql} from eqn. (31) and $\mathbf{Q}^{(i)} \mathbf{Q}^{(j)} \rightarrow Q^2/3$ in very compact form as

$$\begin{aligned} L_{11} &= -2\pi \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{Q}{6} \sum_{l=\pm} \text{sign}(l) \frac{X_{Ql}}{Z_{Ql}^2} \frac{\partial}{\partial \mu} \frac{\partial}{\partial Q} I_{Ql} \\ L_{12} &= -2\pi \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{Q}{6} \sum_{l=\pm} \text{sign}(l) \frac{X_{Ql}}{Z_{Ql}^2} T \frac{\partial}{\partial T} \frac{\partial}{\partial Q} I_{Ql} \\ L_{21} &= -2\pi \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{Q}{6} \sum_{l=\pm} \frac{Y_{Ql}}{Z_{Ql}^2} \frac{\partial}{\partial \mu} \frac{\partial}{\partial Q} I_{Ql} \\ L_{22} &= -2\pi \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{Q}{6} \sum_{l=\pm} \frac{Y_{Ql}}{Z_{Ql}^2} T \frac{\partial}{\partial T} \frac{\partial}{\partial Q} I_{Ql}. \end{aligned} \quad (62)$$

By subtracting the convective part from the energy current, the thermal conductivity for the interacting scalar field is obtained as

$$\lambda = \frac{1}{T} \left(L_{22} - \frac{L_{12}L_{21}}{L_{11}} \right). \quad (63)$$

$L_{11} = d$ is the diffusion coefficient, and in analogy to electromagnetic plasmas we call the quantity

$$\alpha = \frac{1}{T} \frac{L_{21}}{L_{11}} \quad (64)$$

the thermo-force on the charged bosons.

The above formal results can be compared to different calculations on a numerical as well as a formal level. To begin with the latter, we insert a simple spectral function for the boson field at zero chemical potential,

$$\rho_+(E, \mathbf{k}) = \rho_-(E, \mathbf{k}) = \frac{2E\gamma_k}{\pi} \frac{1}{(E^2 - \Omega_k^2)^2 + 4E^2\gamma_k^2} \quad (65)$$

with $\Omega_k^2 = \epsilon_k^2 + \gamma_k^2$ and $\gamma_k \ll \epsilon_k$. Such a spectral function can be considered the lowest order approximation to a system with nontrivial self energy function [17].

In ref. [18], this prescription was applied to the pion dispersion relation in nuclear matter by rewriting the inverse retarded propagator for the pions with real energy E as

$$E^2 - E_\pi^2(\mathbf{k}) - \Pi^R(E, \mathbf{k}) = (E - (\epsilon_k - i\gamma_k)) (E + (\epsilon_k + i\gamma_k)) . \quad (66)$$

With this on-shell approximation the integrals are

$$\begin{aligned} Z_k = \int dE \rho_l(E, \mathbf{k}) &= \frac{1}{2\pi\epsilon_k} \left(\frac{\pi}{2} - \arctan \left(\frac{\gamma_k^2 - \epsilon_k^2}{2\gamma_k\epsilon_k} \right) \right) \\ &\approx \frac{1}{2\epsilon_k} - \frac{\gamma_k}{\pi\epsilon_k^2} + \frac{\gamma_k^3}{3\pi\epsilon_k^4} + \mathcal{O}(\gamma^4) , \end{aligned} \quad (67)$$

and

$$\begin{aligned} \frac{X_{kl}}{Z_{kl}^2} &= \frac{1}{2\pi\gamma_k} + \frac{2}{\pi^2\epsilon_k} + \mathcal{O}(\gamma) \\ \frac{Y_{kl}}{Z_k^2} &= \frac{\epsilon_k}{2\pi\gamma_k} + \frac{2}{\pi^2} + \mathcal{O}(\gamma) . \end{aligned} \quad (68)$$

In the absence of a chemical potential, L_{12} and L_{21} are zero, but diffusion coefficient and thermal conductivity are to lowest order in the width

$$\begin{aligned} d &= -\frac{1}{T} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{3\gamma_k} \frac{\partial}{\partial k} \epsilon_k n(\epsilon_k) (1 + n(\epsilon_k)) \\ \lambda &= -\frac{1}{T} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{k\epsilon_k}{3\gamma_k} \frac{\partial}{\partial k} \epsilon_k n(\epsilon_k) (1 + n(\epsilon_k)) . \end{aligned} \quad (69)$$

Here, $n(\epsilon_k)$ is the local Bose function taken at energy ϵ_k . Apart from the momentum factors, a similar representation for λ was obtained in ref. [8]. The difference can be attributed to the fact that in this reference a hydrodynamical picture was assumed together with $\beta\gamma_k \ll 1$. We thus make a comparison in the high temperature limit, where one obtains

$$\lim_{T \rightarrow \infty} \lambda = \frac{A}{\bar{\gamma}} T^3 . \quad (70)$$

Here, $\bar{\gamma}$ is some momentum averaged width and A a numerical factor. The same limiting behaviour is obtained in [8]. Also other calculations in quantum field theory give qualitatively similar results: The bulk viscosity of matter should rise with the same temperature dependence as the thermal conductivity, and in [19] (and references quoted there) was found to rise as T^3 in a quark-gluon plasma.

The above expressions are also in accordance with naive expectations: The thermal conductivity and diffusion coefficient diverge, if the width of the particles is reduced, i.e., if the interaction is removed. In this case the time needed for the relaxation of a temperature disturbance is infinite.

6 Application to pion propagation in nuclear matter

In the next step one has to look at numerical results. In the spirit of the introductory remarks, pions were chosen for this example. However, to obtain meaningful results one has to go beyond the simple approximation discussed above. Instead of the on-shell approximation of eqn. (66), the so-called Δ -hole model at finite temperature is used [18].

In this model, pions couple to excitations in a gas of (non-relativistic) nucleons and Δ_{33} resonances of width Γ . Although a numerical calculation of the pionic spectral function in this system is not very difficult, for the present calculation we use an analytical approximation

$$\rho_+(E, \mathbf{k}) = \rho_-(E, \mathbf{k}) = \frac{\mathbf{k}^2 C}{\pi} \frac{\Gamma E \omega_\Delta}{(E^2 - \omega_+^2)^2 (E^2 - \omega_-^2)^2 + \Gamma^2 E^2 (E^2 - E_\pi^2)^2} . \quad (71)$$

$E_\pi^2 = \sqrt{\mathbf{k}^2 + m_\pi^2}$ is the free on-shell energy of the pion, ω_Δ and ω_\pm are momentum dependent,

$$\begin{aligned} \omega_\Delta &= E_\Delta(\mathbf{k}) - M_N = \sqrt{\mathbf{k}^2 + M_\Delta^2} - M_N \\ E_{N\Delta} &= \sqrt{\omega_\Delta (\omega_\Delta + g' C)} \\ \omega_\pm^2 &= \frac{1}{2} \left(E_{N\Delta}^2 + (\Gamma/2)^2 + E_\pi^2 \pm \sqrt{(E_{N\Delta}^2 + (\Gamma/2)^2 - E_\pi^2)^2 + 4\mathbf{k}^2 C \omega_\Delta} \right) \end{aligned} \quad (72)$$

and C is the temperature dependent effective coupling constant (see table)

$$C = \frac{8}{9} \left(\frac{f_{N\Delta}^\pi}{m_\pi} \right)^2 \left(\rho_N^0 - \frac{1}{4} \rho_\Delta^0 \right) \quad (73)$$

$f_{N\Delta}^\pi$	g'	m_π	M_N	M_Δ	Γ
2	0.5	0.14 GeV	0.938 GeV	1.232 GeV	0.12 GeV

Table 1: Coupling constants and masses used in the calculations of this work.

with partial densities ρ_N^0 and ρ_Δ^0 of nucleons and Δ 's. The partial densities at a given total baryon density $\rho_B = \rho_N^0 + \rho_\Delta^0$ are obtained by a self-consistent calculation of the chemical potential $\mu_N = \mu_\Delta$ of the fermionic sector.

The above spectral function is obtained with two approximations: First, its is correct only to lowest order in the baryon density (quasistatic approximation), and second it is correct only to lowest order in Γ/M_Δ (asymptotic expansion of the Δ -spectral function for constant Γ). These assumptions yield a polarization tensor of the Δ -nucleon gas

$$\Pi^R(E, \mathbf{k}) = \mathbf{k}^2 C \frac{\omega_\Delta}{(E + i\Gamma/2)^2 - \omega_\Delta^2}, \quad (74)$$

which is then subject to a Migdal (or g') correction (see refs. [18, 20, 21] for more details on this model). Figure 1 is a logarithmic contour plot of the above spectral function at a total baryon density of $\rho_b=1.69$ nuclear matter density and temperature $T = 0.1 \text{ GeV}$. It exhibits two branches, corresponding to the mixing of a coherent Δ -hole excitation and the pion. At low momenta, the strength of the excitations lies predominantly on the lower branch, while at higher momenta the pionic strength has changed to the upper branch.

In figure 2, the product of thermal conductivity λ and temperature T which one obtains with this spectral function is plotted as function of temperature at different densities. Figure 7 is a similar representation of the diffusion coefficient d .

A comparison of these values to *experimental* data is beyond reach for the time being, we thus have to restrict the comparison to other calculations. However, only very few calculations based on field theoretical methods are available. We have already mentioned the agreement with ref. [8] on a formal level, if a simple approximation for the spectral function is inserted (eqn. (65)). Although the structure of the more sophisticated spectral function from (71) is quite different, the high temperature behaviour at the example density of $\rho_b = 1.69 \times$ nuclear matter density inferred from figure 2 is

$$\lambda \approx 9897 \text{ GeV/fm}^2 \times [T/\text{GeV}]^{3.70}, \quad (75)$$

i.e., it rises even faster than estimated by eqn. (70). This is due to the decreasing width of the pion with temperature in the Δ -hole model [18] – a fact, which was not included in eqn. (70). At lower temperatures, the thermal conductivity rises slower with temperature than the asymptotic expression.

The density dependence of the coefficients λ and d is quite small, with the general tendency to have a "stiffer" temperature dependence at lower baryon density. This is again consistent with the fact, that the pion in the Δ -hole model becomes a free particle without the baryonic background.

The exploratory calculation of the present paper was done with a constant width Γ for the Δ -resonance. Even the vacuum spectral function of the Δ is not completely described by such an ansatz, much less its medium dependence

[21, 22]. The width in matter at finite temperature (obtainable by self-consistent calculations currently in progress) might be a factor 2–3 times higher, hence according to eqn. (69) the thermal conductivity should be lower due to the medium dependence of the Δ -width, and its temperature dependence flatter.

For completeness, also *variational* and *hydrodynamical* calculations of the pionic thermal conductivity in the baryon-free regime have to be mentioned [23, 24] (also plotted in figure 2). In these, the thermal conductivity for $T \gtrsim 0.1$ GeV is much lower than obtained in the present paper: Beyond $T \approx 60$ MeV it rises only quadratically with temperature (see thin lines in figure 2 for the corresponding slopes). The results have been obtained by using vacuum pion-pion scattering data as input, and thus the interaction in these models is completely different from the Δ -hole model. Hence, a direct comparison of the results is not justified.

Nevertheless, a short remark on the difference is necessary. Within the hydrodynamic picture a medium effect, and thus also the T^2 -dependence of the thermal conductivity, can be explained by the rise of the effective pion-pion scattering cross-section with temperature [25]. It was already stated, that the medium broadening of the pion might be underestimated in the present paper. However, even when reducing the present results by a factor 2–3 at higher temperatures, we still find that the thermal conductivity of the interacting pions in presence of the hot baryonic background is higher than in refs. [23, 24].

7 Summary and Conclusions

In the present paper, Thermo Field Dynamics (TFD) for inhomogeneous systems [2, 14] is generalized to quantum fields with a continuous single-particle mass spectrum [1]. Following a procedure outlined in refs. [5, 6, 7], the spectral and statistical content of the full two-point function are separated in a well defined approximation. This approximation, i.e., assuming that a *spectral function* still is a meaningful concept, limits the results to situations not too far from equilibrium. However, corrections to this approximation are of second order in the gradients. Furthermore a general formulation in coordinate space is easily achieved [5, 6], which also takes temporal inhomogeneity into account and thus does not trade simplicity for covariance.

By requiring the thermal Bogoliubov symmetry to hold locally in space, a modification of the systems' hamiltonian was obtained. It is well known, that this *gauging* of a global symmetry introduces a minimal coupling to an external classical field, here: A "temperature field" and a "chemical potential field" [15].

In a system close to equilibrium, this coupling was used to calculate transport (or kinetic) coefficients of the interacting quantum field. In contrast to previous derivations [8], neither imaginary time arguments nor perturbation theory in powers of a coupling constant were used for this purpose. Expressions for thermal conductivity and the diffusion coefficient were given in terms of the

equilibrium spectral function of the quantum field. They are dominated by two factors: A Poisson bracket, which stems from a gradient expansion (cf. eqn. (55)), and proportionality to the inverse width of the spectral distribution, cf. eqn. (69).

The Poisson bracket contributes a factor \hbar , while in a naive counting the inverse width contributes a factor $1/\hbar$ to the transport coefficient. This cancellation seems to be the main source for the difficulties one has in obtaining kinetic coefficients from quantum field theory [26, 27].

A different approach to the calculation of transport coefficients is the solution of a Boltzmann equation (e.g. in relaxation time approximation) [23, 28]. However, this approach has the drawback of a zero width quasi-particle approximation at a very early stage of the computation, i.e., it is semi-classical. In contrast to such kinetic models, the formulation of the present work contains *off-shell effects*, and therefore the full quantum description, in a consistent way.

Furthermore, some of the semi-classical calculations retain the relaxation time as an undetermined factor in the final expressions. While TFD in principle also provides an expression for the relaxation time [5, 6], the method presented here is more direct and can be extended systematically.

The new method was tested by calculating thermal conductivity and diffusion coefficient for the pionic component in a hot Δ -nucleon gas. Results were obtained, which are substantially higher than those found with the semi-classical zero width kinetic picture. As was pointed out in the previous section, part of this might be due to the too simplistic picture of the Δ -hole model with constant Δ -width.

However, this finding is also consistent with an observation made, if the Kadanoff-Baym equations are solved directly rather than approximated by a Boltzmann equation: The relaxation of perturbations is slower, if the full quantum description is taken into account [29].

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Figure 1: Pion spectral function of the Δ -hole model at $\rho_b = 1.69$ nuclear density and $T = 0.1$ GeV. Horizontal axis $|\mathbf{k}|$ in GeV, vertical axis E in GeV. Plotted are lines of equal $\rho(E, \mathbf{k})$ on logarithmic scale, see ref. [18] for details.

Figure 2: Pion thermal conductivity \times temperature.
Thin straight lines: slopes $\propto T^3, T^4$
Full thick line: Δ -hole model at $\rho_b = 1.69$ nuclear density;
dashed line 0.92 and dash-dotted line 0.47 nuclear density.
Dash-double-dotted line ref. [24], dotted line ref. [23],
both from $\pi - \pi$ scattering data at $T = 0, \mu_b = 0$.

Figure 3: Pion diffusion coefficient.
Thin straight lines: slopes $\propto T^3, T^4$
Full thick line: Δ -hole model at $\rho_b = 1.69$ nuclear density;
dashed line 0.92 and dash-dotted line 0.47 nuclear density.